Econ 201 Section 5 - Problem Set 1

Due 1/24 by Start of Class

1.

Answer each of the following questions as TRUE, FALSE, or UNCERTAIN, and justify your answer.

(a) If a production function $f(K, L)$ displays constant returns to scale and diminishing marginal product, then $f_{KL} = \frac{\partial^2 f(K,L)}{\partial K \partial L}$ must be positive.

SOLUTION: TRUE Because the production function is CRS, $\forall t > 0$, $f(tK, tL) = tf(K, L)$. Differentiate both sides of the equation with respect to t,

$$f_K(tK, tL)K + f_L(tK, tL)L = f(K, L)$$

If we take $t = 1$, then

$$f_K(K, L)K + f_L(K, L)L = f(K, L)$$

Differentiate with respect to L,

$$f_{KL}(K, L)K + f_{LL}(K, L)L + f_L(K, L) = f_L(K, L)$$

Thus,

$$f_{KL}(K, L) = -\frac{L}{K} f_{LL}(K, L)$$

As the production function satisfies the property of decreasing marginal product, $f_{LL}(K, L) \leq 0$, so

$$f_{KL}(K, L) \geq 0$$

(b) If the firm is cost-minimizing and is at an optimum, the marginal cost is the same whether it changes only labor, only capital, or both.

SOLUTION: TRUE (Assuming the solution to the problem requieres using lagrange and there is not a corner solution). The first order condition of the cost-minimizing problem is $w = \lambda f_L$ and $r = \lambda f_K$. We can rewrite this as $\lambda = \frac{w}{f_L} = \frac{r}{f_K}$ where λ is what it would cost to produce an extra unit of output using only one input. Hence, at the optimum, the marginal cost is the same whether it changes only labor, only capital, or both.

(c) When marginal cost decreases, average cost also decreases. When marginal cost increases, average cost also increases.

SOLUTION: FALSE Average cost increases (decreases) when the marginal cost is larger (smaller) than the average cost. Whether marginal cost is increasing or decreasing does not directly matter to the direction in which average cost will move.

(d) The minimum point of ATC curve is right above the minimum point of AVC curve (at the same y).

SOLUTION: FALSE ATC is the sum of AVC and AFC. At the minimum of ATC, it is neither increasing nor decreasing, which means that the rate of change of the AVC and AFC must balance out. Since the AFC is always decreasing, then AVC must be increasing. Thus, it has to be to the right of the minimum AVC.

$$ATC = AVC + AFC$$

$$ATC = \frac{VC}{y} + \frac{FC}{y}$$

$$\frac{dATC}{dy} = \left[\frac{dVC}{dy}\frac{1}{y} - \frac{1}{y^2}VC\right] - \frac{FC}{y^2} = 0$$

Since $-\frac{FC}{y^2} < 0$, $-\frac{1}{y^2}VC < 0$ and $\frac{1}{y} > 0$, then

$$\frac{dVC}{dy} > 0$$

2.

Consider a firm that has the production function $f(L, K) = L^{1/2} + K^{1/2}$.

(a) Does this production function satisfy monotonicity?

SOLUTION:

$$\frac{d}{dL}f(L,K) = \frac{1}{2}L^{-1/2} \geq 0$$
$$\frac{d}{dK}f(L,K) = \frac{1}{2}K^{-1/2} \geq 0$$

Both marginal products are non-zero for all non-negative values of labor or capital, so monotonicity holds.

(b) Calculate the marginal products of of labor and capital. Does the production function satisfy diminishing marginal product for each input?

SOLUTION:

$$\frac{dMP_L}{dL} = \frac{d}{dL}\frac{1}{2}L^{-1/2} = -\frac{1}{4}L^{-3/2} \leq 0$$
$$\frac{dMP_K}{dK} = \frac{d}{dK}\frac{1}{2}K^{-1/2} = -\frac{1}{4}K^{-3/2} \leq 0$$

The partial derivatives of each marginal products are negative for all non-negative values of labor and capital, so the production function satisfies diminishing marginal product for both inputs.

(c) What is the Technical Rate of Substitution for this firm? Explain the meaning of this in words.

SOLUTION:

$$TRS(L,K) = -\frac{MP_L}{MP_K} = -\frac{L^{-1/2}}{K^{-1/2}} = -\frac{K^{1/2}}{L^{1/2}}$$

It is the amount by which the quantity of capital has to be reduced when one extra unit of labor is used, so that output remains constant.

(d) Does this firm have increasing, constant, or decreasing returns to scale?

SOLUTION:

$$\begin{aligned} f(tL,tK) &= (tL)^{1/2} + (tK)^{1/2} \\ &= t^{1/2}(L^{1/2} + K^{1/2}) \\ &= t^{1/2} f(L,K) \\ &< t f(L,K) \end{aligned}$$

Thus this firm has decreasing returns to scale. Note that we cannot do the trick of summing exponents (which would suggest CRS) as this is not a Cobb-Douglas production function - the factors are being summed rather than multiplied.

Problem 3

Consider the following production functions:

(i) $f(K,L) = K^{1/2}L^{3/4}$

(ii) $f(K,L) = K + L^2$

(iii) $f(K,L) = max\{K,L\}$

Answer the following questions for all three production functions.

(a) Find the marginal products of labor and capital. Are the increasing, constant, or decreasing?

SOLUTION:

(i)

$$\begin{aligned} MP_L &= \frac{d}{dL}K^{1/2}L^{3/4} \\ &= \frac{3}{4}K^{1/2}L^{-1/4} \\ \frac{dMP_L}{L} &= -\frac{3}{16}K^{1/2}L^{-5/4} \leq 0 \\ MP_K &= \frac{d}{dK}K^{1/2}L^{3/4} \\ &= \frac{1}{2}K^{-1/2}L^{3/4} \\ \frac{dMP_K}{K} &= -\frac{1}{4}K^{-3/2}L^{3/4} \leq 0 \end{aligned}$$

Both of the marginal products are decreasing as they are negative on the non-negative range of the inputs.

(ii)

$$\begin{aligned} MP_L &= \frac{d}{dL}K + L^2 \\ &= 2L \\ \frac{dMP_L}{L} &= 2 > 0 \\ MP_K &= \frac{d}{dK}K + L^2 \\ &= 1 \\ \frac{dMP_K}{K} &= 0 \end{aligned}$$

The marginal product of labor is increasing and the marginal product of capital is constant.

(iii)

$$
\begin{aligned}
MP_L &= \frac{d}{dL} max\{K, L\} \\
&= \begin{cases} 1 \; if \; L \geq K \\ 0 \; if \; L < K \end{cases} \\
\frac{dMP_L}{L} &= \begin{cases} 0 \; if \; L > K \\ 0 \; if \; L < K \end{cases} \\
MP_K &= \frac{d}{dK} max\{K, L\} \\
&= \begin{cases} 1 \; if \; K \geq L \\ 0 \; if \; K < L \end{cases} \\
\frac{dMP_K}{K} &= \begin{cases} 0 \; if \; K > L \\ 0 \; if \; K < L \end{cases}
\end{aligned}
$$

The marginal products of both labor and capital are constant (except for the point at which they are equal to one another, in which case they are both undefined).

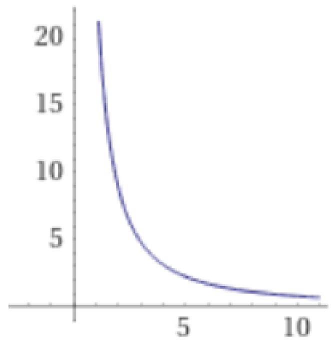
(b) Draw the isoquants. These need not be precise, but should capture the general shape the isoquants would have and have some justification as to why they would possess this shape.

SOLUTION:

(i)

$$
\begin{aligned}
y_0 &= K^{1/2} L^{3/4} \\
K^{1/2} &= y_0 L^{-3/4} \\
K &= y_0^2 L^{-3/2}
\end{aligned}
$$

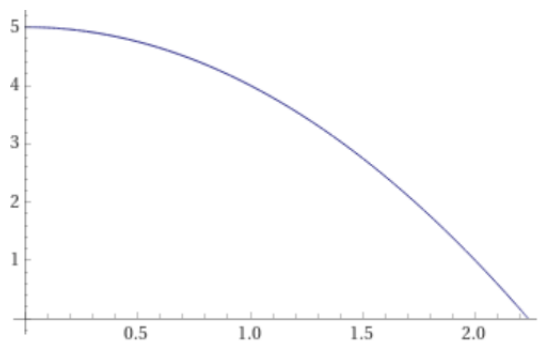
The below depicts this with an arbitrary value of 5 inserted for y_0. Note that this kind of shape could also have been expected based on the fact that this is a Cobb-Douglas production function:



(ii)

$$y_0 = K + L^2$$
$$K = y_0 - L^2$$

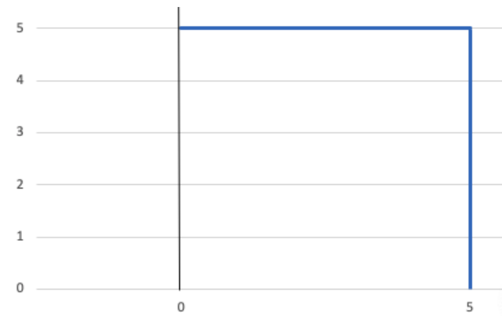
The below depicts this with an arbitrary value of 5 inserted for y_0.



(iii)

$$y_0 = max\{K, L\}$$
$$K \begin{cases} = y_0 \; if \; K > L \\ \leq y_0 \; if \; L = y_0 \geq K \end{cases}$$

If production is equal to the maximum of the inputs, then on an isoquant for any value y_0, the value of the maximum input must be equal to exactly y_0. The lesser input can take on any value from 0 to y_0, inclusive. The below depicts this with and arbitrary value of 5 inserted for y_0.



(c) Is the production function convex?

SOLUTION:

(i) Yes
(ii) No
(iii) No

All of these can also be seen graphically from the isoquants derived in the previous section.

(d) Does the production function display increasing, constant or decreasing returns to scale?

SOLUTION:

(i) Increasing Returns to Scale

$$\begin{aligned} f(tL, tK) &= (tK)^{1/2}(tL)^{3/4} \\ &= t^{1/2}t^{3/4}K^{1/2}L^{3/4} \\ &= t^{5/4}K^{1/2}L^{3/4} \\ &> tK^{1/2}L^{3/4} = tf(K, L) \end{aligned}$$

Note that this is a Cobb-Douglas production function, so we could also have known this by summing this exponents, and seeing that their sum is greater than 1.

(ii) Increasing Returns to Scale

$$\begin{aligned} f(tL, tK) &= tK + (tL)^2 \\ &= t(K + tL^2) \\ &> t(K + L^2) = tf(K, L) \end{aligned}$$

(iii) Constant Returns to Scale

$$\begin{aligned} f(tL, tK) &= max\{tK, tL\} \\ &= tmax\{K, L\} \\ &= tf(K, L) \end{aligned}$$

If all inputs are scaled up by t this implies that which input is the maximum is scaled up by t. Thus, no matter what the original configuration of inputs, we know that the original output will be scaled up by t.

Problem 4

Consider a farmer with the following production function that uses labor L, capital K, and land N:

$$f(L, K, N) = L^{1/3}K^{1/6}N^{1/2}$$

This farmer faces costs for his input w for labor, r for capital, and p for land.

(a) What are the returns to scale for this production function?

SOLUTION: Note that this production function still has the form of a Cobb-Douglas production function - inputs are all multiplied by one another with exponents. Using our Cobb-Douglas trick, we see that the exponents sum to 1, implying CRS. We can also verify this by inserting scaled versions of each input into the production function.

(b) In the long run, the farmer can adjust all three inputs freely. Formally state the cost minimization problem if the farmer wants to produce a certain amount $\overline{y}$, and solve the long run cost minimization problem, finding the conditional demand for each input as well the long run minimum cost function.

SOLUTION: We can use the Lagrangian method to solve this problem:

$$\min_{\{L,K,N\}} C(L,K,N) = wL + rK + pN$$
$$s.t.\ L^{1/3}K^{1/6}N^{1/2} = \overline{y}$$
$$\mathcal{L} = wL + rK + pN + \lambda(\overline{y} - L^{1/3}K^{1/6}N^{1/2})$$
$$[\text{L}] : w - \frac{1}{3}\lambda L^{-2/3}K^{1/6}N^{1/2} = 0$$
$$[\text{K}] : r - \frac{1}{6}\lambda L^{1/3}K^{-5/6}N^{1/2} = 0$$
$$[\text{N}] : p - \frac{1}{2}\lambda L^{1/3}K^{1/6}N^{-1/2} = 0$$
$$[\lambda] : \overline{y} - L^{1/3}K^{1/6}N^{1/2} = 0$$

Combine the first and second FOCs

$$\frac{w}{r} = \frac{2K}{L}$$
$$L = \frac{r}{w}2K$$

Plug in $K = \overline{y}^6L^{-2}N^{-3}$ from the third FOC.

$$L = (2\frac{r}{w})^{1/3}\overline{y}^2N^{-1}$$

Now combine the first and third FOCs to get rid of the N.

$$\frac{w}{p} = \frac{2N}{3L}$$
$$N = \frac{3w}{2p}L$$
$$L = (2\frac{r}{w})^{1/3}\overline{y}^2(\frac{3w}{2p}L)^{-1}$$
$$L^*(w,r,p,\overline{y}) = (2\frac{r}{w})^{1/6}(\frac{2p}{3w})^{1/2}\overline{y}$$

Finally, plug this into our expressions for N and K to get the conditional demands for land and capital, and combine all three into a minimum cost function.

$$N^*(w,r,p,\overline{y}) = (2\frac{r}{w})^{1/6}(\frac{2p}{3w})^{-1/2}\overline{y}$$
$$K^*(w,r,p,\overline{y}) = (2\frac{r}{w})^{-5/6}(\frac{2p}{3w})^{1/2}\overline{y}$$
$$C^*(w,r,p,\overline{y}) = w(2\frac{r}{w})^{1/6}(\frac{2p}{3w})^{1/2}\overline{y} + r(2\frac{r}{w})^{-5/6}(\frac{2p}{3w})^{1/2}\overline{y} + p(2\frac{r}{w})^{1/6}(\frac{2p}{3w})^{-1/2}\overline{y}$$

Note that we can check our conditional demands by noting that if we plug each value into the production function, we get back $\overline{y}$.

(c) In the short run, the farmer has a fixed amount of land $\overline{N}$ that he cannot vary. Solve the short run cost minimization problem, again finding the conditional demand for each (free) input as well the short run minimum cost function.

SOLUTION: We can use the Lagrangian method to solve this problem:

$$\min_{\{L,K\}} C(L,K,\overline{N}) = wL + rK + p\overline{N}$$
$$s.t.\ L^{1/3}K^{1/6}\overline{N}^{1/2} = \overline{y}$$
$$\mathcal{L} = wL + rK + p\overline{N} + \lambda(\overline{y} - L^{1/3}K^{1/6}\overline{N}^{1/2})$$
$$[\text{L}] : w - \frac{1}{3}\lambda L^{-2/3}K^{1/6}\overline{N}^{1/2} = 0$$
$$[\text{K}] : r - \frac{1}{6}\lambda L^{1/3}K^{-5/6}\overline{N}^{1/2} = 0$$
$$[\lambda] : \overline{y} - L^{1/3}K^{1/6}\overline{N}^{1/2} = 0$$

Combine the first and second FOCs

$$\frac{w}{r} = \frac{2K}{L}$$
$$L = \frac{r}{w}2K$$

Plug in $K = \overline{y}^6 L^{-2}\overline{N}^{-3}$ from the third FOC.

$$L^*(w,r,\overline{N},\overline{y}) = (2\frac{r}{w})^{1/3}\overline{y}^2\overline{N}^{-1}$$

Finally, plug this into our expressions just K to get the last conditional demand (remember land is fixed), and combine both into a minimum cost function.

$$K^*(w,r,\overline{N},\overline{y}) = (2\frac{r}{w})^{-2/3}\overline{y}^2\overline{N}^{-1}$$
$$C^*(w,r,p,\overline{y}) = w(2\frac{r}{w})^{1/3}\overline{y}^2\overline{N}^{-1} + r(2\frac{r}{w})^{-2/3}\overline{y}^2\overline{N}^{-1} + p\overline{N}$$

Note that we can check our conditional demands by noting that if we plug each value into the production function, we get back $\overline{y}$.

(d) What input costs appear in the short run conditional demand functions? What input costs appear in the long run conditional demand functions? Explain in words why the two are either the same or different.

SOLUTION: We see that all three input prices appear in the long run function, but only w and r appear in the short run function. n goes away in the short run because land is fixed, and so the price of land is irrelevant to cost minimization because the amount of land cannot adjust to the price of land. The other two remain in, because the relative prices of labor and capital can still cause adjustments between the amounts of those two inputs.

(e) Under what conditions will the short run minimum cost be the same as the long run minimum cost?

SOLUTION: The short run minimum cost will equal the long run minimum cost if $\overline{N}$ is equal to the long run optimal amount of land, $N^*(w,r,p,\overline{y}) = (2\frac{r}{w})^{1/6}(\frac{2p}{3w})^{-1/2}\overline{y}$.

Problem 5

Consider a firm with the production function $f(L,K) = Kln(L)$. For the rest of this problem, assume that we are in the short run, so there is a fixed value of capital $\overline{K}$.

(a) What is the minimum (total) cost function?

SOLUTION: In the short run, we can find the conditional demand for labor directly from the constraint, and then plug this and our fixed value of capital into a cost function to get minimum (total) cost.

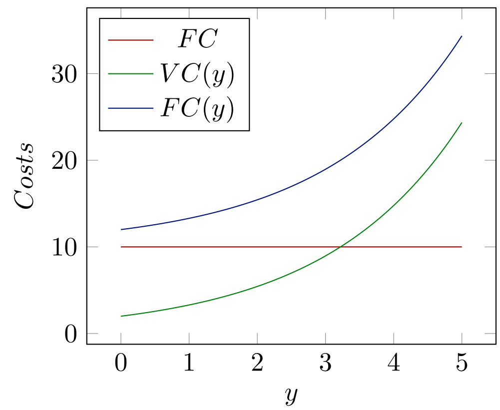
$$\begin{aligned}
\overline{K}ln(L) &= \overline{y} \\
ln(L) &= \frac{\overline{y}}{\overline{K}} \\
L^*(\overline{K},\overline{y}) &= e^{\frac{\overline{y}}{\overline{K}}} \\
C^{SR}(\overline{K},\overline{y}) &= r\overline{K} + we^{\frac{\overline{y}}{\overline{K}}}
\end{aligned}$$

(b) Say that $w = 2$, $r = 5$, and $\overline{K} = 2$. What are the fixed cost and variable costs, as functions of output?

SOLUTION:

$$\begin{aligned}
FC &= 10 \\
VC(y) &= 2e^{\frac{y}{2}}
\end{aligned}$$

(c) Draw total cost, fixed cost, and variable cost on the same graph.



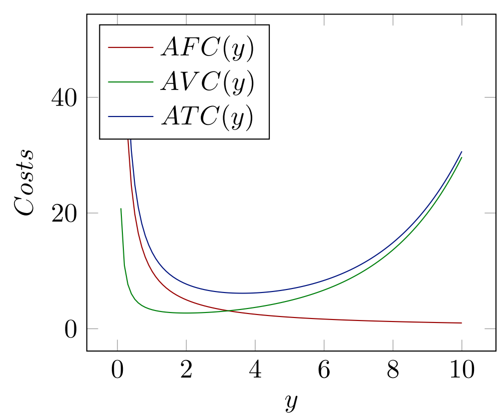
SOLUTION:

(d) Find expressions for the AFC, AVC, and ATC in terms of output.

SOLUTION:

$$AFC(y) = \frac{10}{y}$$
$$AVC(y) = \frac{2e^{y/2}}{y}$$
$$ATC(y) = \frac{10 + 2e^{y/2}}{y}$$

(e) Draw AFC, AVC, and ATC on the same graph.



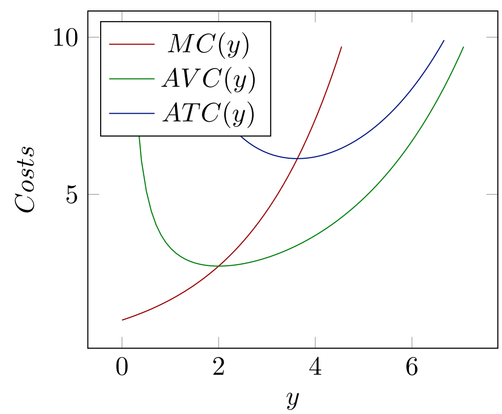
SOLUTION:

(f) Find an expression for the marginal cost in terms of output.

SOLUTION:

$$MC(y) = \frac{d}{dy} 10 + 2e^{y/2}$$
$$MC(y) = e^{y/2}$$

(g) On a separate graph, draw the ATC, AVC, and MC. What do we know about the intersection of MC with each of the other two curves? Why is this the case (explain in words not math)?



SOLUTION: The MC will intersect both the AVC and the ATC at their minima. This occurs because if the MC is below either one, this indicates that the next unit produced will cost less than average total (variable) cost, and will therefore cause the total (variable) cost to decrease. If the MC is above either one, the next unit produced will cost more than average total (variable) cost, and will therefore cause the total (variable) cost to increase. Therefore, at the point where they meet, the AVC and ATC are both neither increasing nor decreasing. Given that they are both convex, this indicates that this is the minima of the functions.

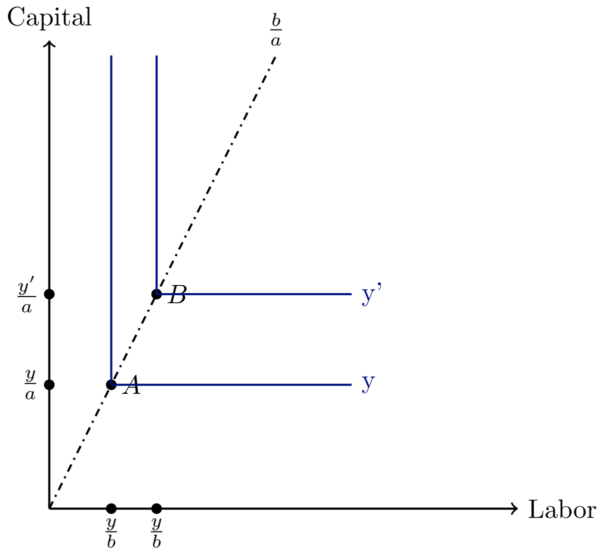
Problem 6

You are the owner of three shoveling companies. Each one of these companies has a perfect complements technology that uses labor (L) and shovels (K) to produce the holes (y).

(a) The perfect complements technology can be expressed as: $f(K, L) = \min\{aK, bL\}$ for $a, b > 0$. For any two arbitrary levels of output y and y' ($y' > y$), draw the two isoquants explicitly showing the roles of a, b, y and y' on the two axes.

SOLUTION:

Perfect complements



As can be seen in the plot above, when the production function takes the form $y = min\{aK, bL\}$ then the isoquates are L-shaped. At point A amounts of labor and capital are such that $aK = bL = y$, while at point B, $aK' = bL' = y'$ and $y' > y$. Parameters a, b determine the ratio in which inputs are used. At every optimal point (the vertices of the 'L's) we have that $aK = bL \to K = \frac{b}{a}L$, which is the dotted line plotted.

(b) More specifically, the technology faced by the three companies are given by:

$$f_1(K, L) = y_1 = [\min\{K, L\}]^{1/2}$$
$$f_2(K, L) = y_2 = \min\{K, L\}$$
$$f_3(K, L) = y_3 = [\min\{K, L\}]^2$$

For each one of the companies, explain whether it is facing a CRS, IRS or DRS.

SOLUTION: For the first firm,

$$\begin{aligned} f_1(tK, tL) &= min\{tK, tL\}^{\frac{1}{2}} = [tmin\{K, L\}]^{\frac{1}{2}} = t^{\frac{1}{2}}[min\{K, L\}]^{\frac{1}{2}} \\ &= t^{\frac{1}{2}} f_1(K, L) < t f_1(K, L) \end{aligned}$$

Therefore f_1 exhibits decreasing returns to scale.

For the second firm,

$$f_2(tK, tL) = min\{tK, tL\} = [tmin\{K, L\}] = tf_2(K, L)$$

Therefore f_2 exhibits constant returns to scale.

For the third firm,

$$\begin{aligned} f_3(tK, tL) &= min\{tK, tL\}^2 = [tmin\{K, L\}]^2 = t^2[min\{K, L\}]^2 \\ &= t^2 f_3(K, L) > tf_1(K, L) \end{aligned}$$

Therefore f_3 exhibits increasing returns to scale.

(c) For simplicity, let $r = w = 1$. For each company, solve the cost minimization problem and find $K^*(y), L^*(y), C^*(y)$. (NB: we have removed r and w from the function's arguments as we have assigned numerical values to them, and for simplicity) (Hint: No need to use the Lagrangian method. Just use your economic intuition.)

SOLUTION: In order to minimize costs given a desired output there should be no wasted inputs. Thereform firms will always choose $K = L : y = f(K, L)$.

For company 1,this means:

$$y_1 = \sqrt{K_1} = \sqrt{L_1} \rightarrow L_1 = K_1 = y_1^2$$

And therefore total costs are $TC^1(y) = 2y^2$. For company 2, this means:

$$y_1 = K_1 = L_1 \rightarrow L_1 = K_1 = y_2$$

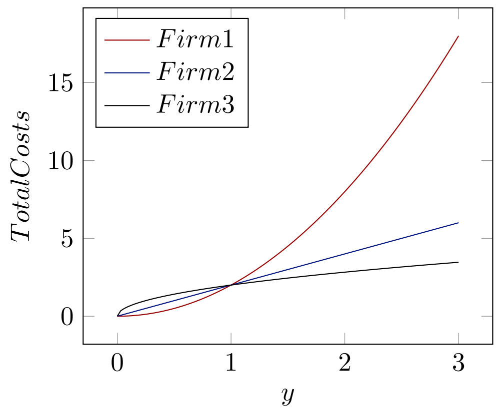
And therefore total costs are $TC^2(y) = 2y$. For company 3, this means:

$$y_1 = K_1^2 = L_1^2 \rightarrow L_1 = K_1 = \sqrt{y_3}$$

And therefore total costs are $TC^3(y) = 2\sqrt{y}$.

(d) This part helps you understand how the returns to scale characteristic of the production function (from part b) relates to the shape of the cost function. Placing costs on the y-axis and output on the x-axis, draw the cost functions of each company from part c). You don't need to be precise but just capture the overall shape of the functions.

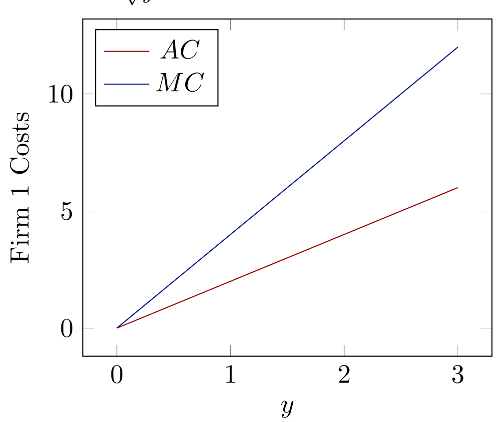
SOLUTION: Here we plot the total cost functions for the three firms:

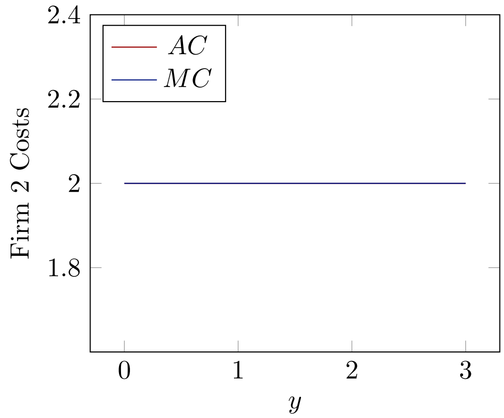


As should be intuitive, the firm with increasing returns (Firm 3) has lower costs when desired production is high, but higher costs when desired production is lower.

(e) Similarly, for each company, find the expression for the marginal cost and average total cost curves. Then, draw them (do them in three different graphs to avoid any confusion). In this part, we will learn how the shape of the cost function affects the shapes of the ATC and MC curves.

SOLUTION: For Firm 1, $ATC(y) = 2y$ and $MC(y) = 4y$. For Firm 2, $ATC(y) = 2$ and $MC(y) = 2$. For Firm 3, $ATC(y) = \frac{2}{\sqrt{y}}$ and $MC(y) = \frac{1}{\sqrt{y}}$. Plotting average and marginal costs for each firms:




2.4
2.2
2
1.8
Firm 2 Costs
0
1
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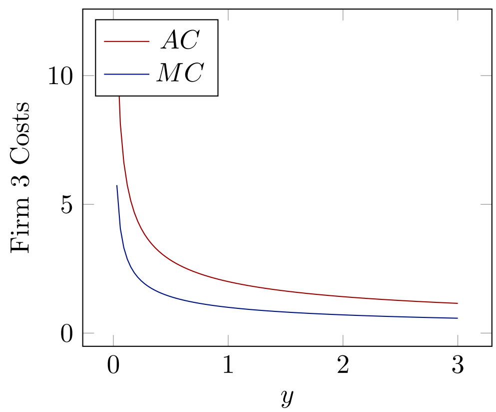
3
y
AC
MC


10
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Firm 3 Costs
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