Econ 201 Section 5 - Problem Set 4 Solutions

Problem 1

Say that the city of Chicago grants you monopoly power in selling textbooks to college students. Your store serves two equally sized client groups - UChicago students and Northwestern students. Suppose the quantities demanded by these two groups are given by:

$$y^c = 10 - p^c$$
$$y^n = a - p^n$$

where $a < 10$. Assume that the marginal cost of textbook production is 0.

(a) Suppose you are restricted to setting one price. Calculate the price you should charge to maximize your profit as well as the quantity sold for any value of $a < 10$. Hint: There will be a kink in total demand at $p = a$ (because Northwestern students cannot demand a negative number of books). It's best to solve separately for the best prices above and below a.

SOLUTION: If we set a price above a, Northwestern students will demand 0 books, so demand will be fully captured by UChicago students for such prices. We thus maximize first just with respect to this group:

$$\max_y = (10 - y)y$$
$$[y]: \ 10 = 2y$$
$$\Rightarrow y = 5$$
$$\Rightarrow p = 5$$
$$\Rightarrow \pi = 25$$

For a price below a, we see that total demand will be equal to the sum of the demands of both groups:

$$y^{tot} = 10 + a - 2p^{tot}$$
$$\Rightarrow p^{tot} = \begin{cases} 10 - y^{tot} \ if \ p > a \\ \frac{10+a}{2} - \frac{y^{tot}}{2} \ if \ p \leq a \end{cases}$$

We can use the second case of the inverse demand curve there to maximize profit for a price that is less than a:

$$\max_y = (\frac{10+a}{2} - \frac{y^{tot}}{2})y$$
$$[y]: \; \frac{10+a}{2} = y$$
$$\Rightarrow y = \frac{10+a}{2}$$
$$\Rightarrow p = \frac{10+a}{4}$$
$$\Rightarrow \pi = \frac{(10+a)^2}{8}$$

The final step is to now determine whether we want to set price above a and only sell to UChicago students or set a lower price and sell to both groups. We can do this by comparing optimal profits across the two cases:

$$25 \lesseqgtr \frac{(10+a)^2}{8}$$
$$\sqrt{200} - 10 \lesseqgtr a$$
$$4.14 \approx \lesseqgtr a$$

The profits equalize at (approximately) a value of $a = 4.14$. Thus, for values of a greater than that, we sell to both groups and for values of a less than that, we only sell to UChicago students.

(b) Suppose now that you can set two different prices to each group. Calculate the price charged and quantity sold to each group.

SOLUTION: This is equivalent to solving the monopoly problem within each group. We can immediately note that the profit maximizing solution for UChicago students will be the same as the gneral problem for a price above a, as we discussed above, so we know:

$$\Rightarrow y^c = 5$$
$$\Rightarrow p^c = 5$$
$$\Rightarrow \pi^c = 25$$

Now all that's left to do is solve the equivalent problem for just Northwestern students:

$$\begin{aligned}\max_{y^n} &= (a - y^n)y^n \\ [y^n]:\ a &= 2y^n \\ \Rightarrow y^n &= \frac{a}{2} \\ \Rightarrow p^n &= \frac{a}{2} \\ \Rightarrow \pi^n &= \frac{a^2}{4}\end{aligned}$$

(c) How do the profits and quantities compare between parts a) and b). Explain the intuition for these results. Say that you gain the ability to read minds and can charge each student at their exact willingness to pay. How would profit and quantity in that situation compare to parts a) and b)? Why?

SOLUTION: In the event that the optimal solution in part a) is to not sell to Northwestern students, then introducing price discrimination will clearly increase both profit and total sales. This is a direct consequence of price discrimination "allowing" for sales to Northwestern students, which was otherwise non-optimal.
We see that in the other case, the total quantity is actually the same. Comparing profits:

$$\begin{aligned}\frac{(10+a)^2}{8} &\lesseqgtr 25 + \frac{a^2}{4} \\ 20a &\lesseqgtr 100 + a^2\end{aligned}$$

$a < 10$ implies that the right-hand side of the above is always greater, so price discrimination will lead to greater profits. This is not surprising, as allowing for price discrimination allows the firm to more precisely target prices to customers' willingness-to-pay, creating room for greater optimization of profit. In general, price discrimination should always weakly increase profit, as a firm that *can* price discriminate could always just charge a uniform price, if that was the optimal thing to do.

Problem 2

(a) Solve for all pure and mixed strategy Nash equilibria of the following two games (using any method you like). (Hint: the second game has infinitely many Nash equilibria.)

		Player 2	
		L	R
Player 1	T	$(6, 0)$	$(0, 6)$
	B	$(3, 2)$	$(6, 0)$

		Player 2	
		L	R
Player 1	T	$(0, 1)$	$(0, 2)$
	B	$(2, 2)$	$(0, 1)$

SOLUTION: One way to solve this problem is to consider the best response functions. For the first game, we start by considering P1's expected utilities for T and B given P2's strategy of playing L with probability q:

$$u_1(T, q) = 6q + 0(1 - q)$$
$$u_1(B, q) = 3q + 6(1 - q)$$
$$\Rightarrow u_1(T, q) \geq u_1(B, q) \Leftrightarrow q \geq \frac{2}{3}$$

From the above, we can infer that P1 will play T for any $q > 2/3$, will play B for any $q > 2/3$, and is indifferent and therefore willing to play anything at $q = 2/3$. We can now turn to P2's best responses, in a similar way, considering their utilities for L and R in response to P1 playing L with probability p:

$$u_2(L, p) = 0p + 6(1 - p)$$
$$u_2(R, p) = 2p + 0(1 - p)$$
$$\Rightarrow u_2(L, p) \geq u_1(R, p) \Leftrightarrow p \leq \frac{1}{4}$$

and so, P2 will play L for any $p < 1/4$, play R for any $p > 1/4$, and is willing to play anything at $p = 1/4$. Putting this all together (drawing the best response curves on a graph may help here), we get that there is only one equilibrium, where $q = 2/3$ and $p = 1/4$, which we can express as $((\frac{1}{4}, \frac{3}{4}), (\frac{2}{3}, \frac{1}{3}))$.

Let's move on to the second game, and use the same method to solve it. Start with P1 and compare utilities for P2 playing L with probability q:

$$u_1(T, q) = 0q + 0(1 - q)$$
$$u_1(B, q) = 2q + 0(1 - q)$$

Here P1 will strictly prefer B except in the specific case that P2 plays R as a pure strategy, in which case P1 is indifferent. Turning to P2's utilties for P1 playing L with probability p:

$$u_2(L,p) = 1p + 2(1-p)$$
$$u_2(R,p) = 2p + 1(1-p)$$
$$\Rightarrow u_2(L,p) \geq u_1(R,p) \Leftrightarrow p \leq \frac{1}{2}$$

So P2 will play L for any $p < 1/2$, will play R for any $p > 1/2$, and is willing to play anything at $p = 1/2$.

Thus, there are two pure strategy NE's (underlining them in the normal form is the easiest way to see this), at (T,R) and (B,L). The former is slightly interesting, as it is sustained by the fact that P1 is indifferent if P2 chooses R. However, this same indifference means that P1 is willing to play any strategy if P2 chooses R, and we know P2 will (be willing to) choose R for any $p \geq 1/2$. Thus, there are infinite mixed strategy NE of the form ((p,(1-p)),R) for any $p \geq 1/2$.

(b) In the following game, we will use both the concepts of "mixed strategy" and "strictly dominated strategy".

		Player 2		
		L	M	R
Player 1	T	$(6,6)$	$(1,2)$	$(3,3)$
	C	$(2,1)$	$(4,7)$	$(4,3)$
	B	$(3,4)$	$(2,5)$	$(3,9)$

(i) For player 1, show that mixing two of his pure strategies strictly dominates the third pure strategy.

SOLUTION: We want to show that playing two of the strategies, each with some probability, will dominate the third strategy. Looking at the table, we see that B is never the best action, although it is not dominated by either T or C alone. We need to find a probability p of playing T (and probability $1-p$ of playing C) such that the mixture of T and C will dominate B, i.e., that the following three inequalities will hold:

$$[s_2 = L] : \; p6 + (1-p)2 > 3$$
$$[s_2 = M] : \; p1 + (1-p)4 > 2$$
$$[s_2 = R] : \; p3 + (1-p)4 > 3$$

There are a wide variety of values that will satisfy these three inequalities - any value of $p \in (\frac{1}{4}, \frac{2}{3})$. As long as we identify at least one of them, we know that P1 has at least one strategy that strictly dominates B, so P1 will never want to play B, and we can eliminate B from the game in our search for equilibria.

(ii) After eliminating the dominated strategy in part (i), show that for player 2 also mixing two pure strategies strictly dominates the third pure strategy. Eliminate it.

SOLUTION: We now doe a similar thing for P2 considering the game without B in it. This time, we see that R is never the maximum utility for P2, although, again, neither L or M will dominate it alone. So, we need to find a probability q of playing L (and $1-q$ of playing M) that will satisfy the following inequalities:

$$[s_1 = T]: \; q6 + (1-q)2 > 3$$
$$[s_1 = C]: \; q1 + (1-q)7 > 3$$

Again there are many values of q that will work here - any on the range $q \in (\frac{1}{4}, \frac{2}{3})$ will do. Once we identify even one such probability, we know that P2 has a strategy that will dominate R in the (reduced) game, and so P2 will never want to play R (given that they know that P1 will never want to play B), and we can eliminate R from the game.

(iii) Find both the pure and mixed strategy NE for the residual game.

SOLUTION: After eliminating B and R, we are left with a 2x2 normal form game. If we underline the best responses, we find two pure NEs, (T,L) and (C,M): To consider any mixed NEs we now want to

		Player 2	
		L	M
Player 1	T	$(\underline{6}, \underline{6})$	$(1, 2)$
	C	$(2, 1)$	$(\underline{4}, \underline{7})$

consider any strategies that will induce indifference for the players. Let p be the probability that P1 plays T and q be the probability that P2 plays L. Then, P1 is indifferent between her actions, and willing to mix, if:

$$u_1(T, q) = u_1(C, q)$$
$$q6 + (1-q)1 = q2 + (1-q)4$$
$$q = \frac{3}{7}$$

Thus, P1 is willing to mix if and only if $q = \frac{3}{7}$. Similarly, P2 is willing to mix between her actions, if

$$u_2(L, p) = u_2(M, p)$$
$$p6 + (1 - p)1 = p2 + (1 - p)7$$
$$p = \frac{3}{5}$$

And so P2 is willing to mix if and only if $p = \frac{3}{5}$. Thus, along with the two pure NEs we found earlier, there is exactly one mixed NE at $(\frac{3}{5}, \frac{2}{5}), (\frac{3}{7}, \frac{4}{7})$.

Problem 3

N bidders are bidding in an auction for one indivisible object. Bidder i has a private value $v_i < 100$ for the object. Each of them can placed a sealed bid (For simplicity, assume they can only bid in whole dollars and cannot bid over 100). The highest bid wins. Winner pays the second highest bid. Losers pay nothing and get nothing. If there are multiple winners, they evenly share the object and the payment.

(a) Who are the players in this game?

SOLUTION: The players are the bidders in the auction, $\{1, ..., N\}$.

(b) What is the strategy set for each player? (i.e. What are the choices each player can choose from?)

SOLUTION: The strategies are the bids that each player can choose. We see here that each player can bid up to 100, obviously cannot bid below 0, and must pay in whole dollars. Thus, for each bidder i, the strategy space is:

$$b_i \in \{0, 1, ..., 99, 100\}$$

(c) Describe the payoff function for each player. (It would be a function of the actions of all agents.)

SOLUTION: If a bidder wins outright, they get the value that they have bid minus the amount of dollars they pay, which we know is equal to the second highest bid. If the bidder ties, they split the value and the payment. If they bid anything other than the highest bid, they just get 0. Thus, if w is the number of winning bids, the payout function for any bidder i will be:

$$u_i(b_1, b_2, ..., b_N) = \begin{cases} \frac{1}{w}(v_i - \max_{i \neq j} b_j) \ if \ b_i = \max\{b_1, ..., b_N\} \\ 0 \ if \ b_i \neq \max\{b_1, ..., b_N\} \end{cases}$$

(d) Define a Nash Equilibrium in this game. Note you do not need to actually solve for the NE - merely denote the circumstances that would be an NE for this game.

SOLUTION: The NE for this game, as with any game, will be the place where, for each player i, the optimal strategy, given that the other players are playing the NE strategy, will be to also play the NE strategy. Using the notation we have established, for this game that would mean $(b_1^*, b_2^*, ..., b_N^*)$ is an NE if, for all players i, it is the case that:

$$u_i(b_i^*, b_{-i}^*) \geq u_i(b_i, b_{-i}^*) \; \forall \; b_i \in \{1, ..., N\}$$

The question only asks for the definition of an NE, as above, but, for the curious, the NE for this type of auction will occur where every player plays their true value of the object, v_i.

Problem 4

In class we mainly studied the static games where the action set for each agent includes finite discrete actions. Sometimes the action set can be infinite and continuous. Consider the following example. Two firms are selling to a market with a fixed market demand. Market price is given by the market demand

$$P(Q) = \begin{cases} a - bQ & \text{if } Q \leq \frac{a}{b} \\ 0 & \text{if } Q > \frac{a}{b} \end{cases}$$

where Q is the sum of the two firms' outputs. Each of them chooses a quantity to produce at a constant marginal cost $c < a$. Each firm wishes to maximize its own profits.

(a) Who are the agents in this game?

SOLUTION: The agents are the two firms.

(b) What is the strategy set for each player? (i.e.What are the choices each player can choose from?)

SOLUTION: Each firm is choosing a quantity. Hypothetically, each firm can choose any positive number to produce, so for both firms, the strategy set is:

$$q_i \in [0, \infty)$$

(c) Describe the payoff function for each player. (It would be a function of the actions of all agents.)

SOLUTION: The payoffs are profits. Because both firms sell to the same market, the price the firms received will depend on the actions of both firms, and, therefore, so will the profits. This is why this situation constitutes a "game" - both agents must take the other's actions into account. The profit will specifically look like, for both firms:

$$\pi_1(q_1, q_2) = \begin{cases} (a - b(q_1 + q_2))q_1 - cq_1 \; if \; (q_1 + q_2) \leq \frac{a}{b} \\ 0 \; if \; q_1 + q_2) > \frac{a}{b} \end{cases}$$

$$\pi_2(q_1, q_2) = \begin{cases} (a - b(q_1 + q_2))q_2 - cq_2 \; if \; (q_1 + q_2) \leq \frac{a}{b} \\ 0 \; if \; q_1 + q_2) > \frac{a}{b} \end{cases}$$

(d) Define a Nash Equilibrium in this game. Note you do not need to actually solve for the NE - merely denote the circumstances that would be an NE for this game.

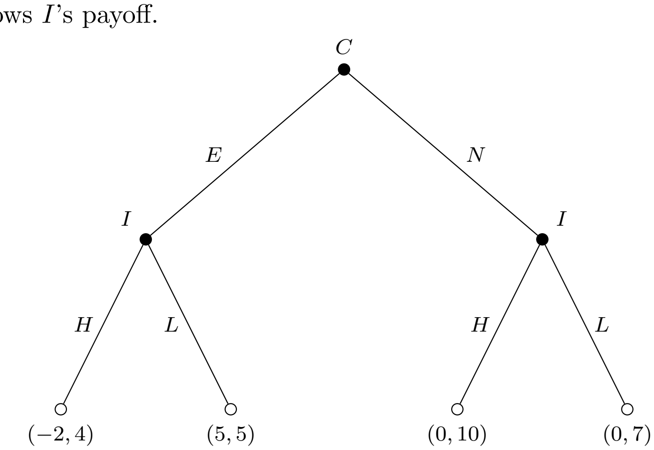
SOLUTION: the NE will be where both firms are simultaneously supplying the optimal quantity to the market, given what the other firm is doing. Thus, we can say that (q_1^*, q_2^*) is an NE if,

$$\pi_1(q_1^*, q_2^*) \geq \pi_1(q_1, q_2^*) \; \forall \; q_1 \in [0, \infty)$$

$$\pi_2(q_1^*, q_2^*) \geq \pi_2(q_1^*, q_2) \; \forall \; q_2 \in [0, \infty)$$

Problem 5

Consider two firms: an incumbent (I) and a potential competitor (C). First, the potential competitor has to decide whether to enter the market (E) or not enter the market (N), and then the incumbent has to decide whether to produce a high quantity (H) or low quantity (L). This game has the following extensive form. The first number in each payoff pair is C's payoff. The second number shows I's payoff.



(a) Write down the set of strategies for each player, and the normal form of this game.

SOLUTION: The strategies for the competitor are simple, enter or not enter,

$$S_C = \{E, N\}$$

For the incumbent, to fully characterize the set of strategies, we need to consider contingent plans - for each move the competitor could make, the incumbent needs to have a response. Say that in each strategy for the incumbent, the first element is the response to the competitor entering and the second element is the response to the competitor not entering, and then we can denote this:

$$S_I = \{(H, H), (H, L), (L, H), (L, L)\}$$

We can combine all of these strategies, along with the payoffs in a normal form:

		Incumbent			
		(H, H)	(H, L)	(L, H)	(L, L)
	E	$(-2, 4)$	$(-2, 4)$	$(5, 5)$	$(5, 5)$
Competitor	N	$(0, 10)$	$(0, 7)$	$(0, 10)$	$(0, 7)$

(b) Find the pure strategy Nash Equilibrium/a of this game.

SOLUTION: To find all of the pure strategy NE's, we can underline the best responses in the normal form we put together above: There are three

		Incumbent			
		(H, H)	(H, L)	(L, H)	(L, L)
	E	$(-2, 4)$	$(-2, 4)$	$(\underline{5}, \underline{5})$	$(\underline{5}, \underline{5})$
Competitor	N	$(\underline{0}, \underline{10})$	$(\underline{0}, 7)$	$(0, \underline{10})$	$(0, 7)$

cells with two underlines corresponding to three pure strategy NEs:

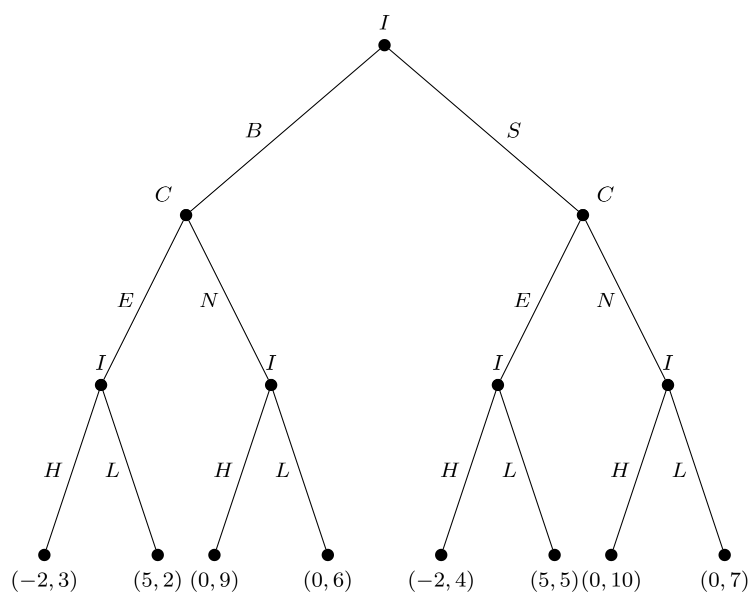
$$\{[N, (H, H)]; [E, (L, H)]; [E, (L, L)]\}$$

(c) How many subgames does the main game have? What is/are the subgame perfect Nash equilibrium/a (SPNE)? Explain why some (if any) of the pure strategy NE are/is not SPNE.

SOLUTION: Because there is perfect information, each node corresponds to a subgame. There are three nodes, and so there are three subgames (including the very top node which corresponds to the entire game as a subgame of itself).

Using backwards induction, we can see that if the competitor enters, the incumbent wants to choose L and if the competitor does not enter, the incumbent wants to choose H. $[N, (H, H)]$ violates the former and $[E, (L, L)]$ violates the latter, so neither is an SPNE. $[E, (L, H)]$ has the incumbent doing the "right" thing for both of the competitor's options, and so it is, indeed, an SPNE.

(d) Consider the following change to the above game. Before the potential competitor makes a move, the incumbent can decide to build a big factory (B) or a small factory (S). The first number in each payoff pair is C's payoff. The second number shows I's payoff. The extensive form of the new game is:



In the above game, a strategy for C is written as (i, j) where i is the action when I plays B and j is the action when I plays S. Similarly, a strategy for I is written as $[v, (w, x), (y, z)]$ where v is the action in the initial node, (w, x) are the actions when C plays E and N respectively after I has played B (top branch), and (y, z) are the actions when C plays E and N respectively after I has played S (bottom branch). Using the above extensive form and backward induction, find all the SPNE (if any). Don't get confused with the payoffs.

SOLUTION: To find the SPNE's, we again use backwards induction. We start with the bottom "layer" of nodes - the four nodes that the incumbent has after the competitor has moved. Going from left to right, at each of those nodes, the incumbent would want to choose H, H, L, and H respectively. Assume the competitor knows this and move "up" to the competitor's two nodes. From left to right, the competitor will choose N and E. Assume the incumbent knows all of this and move up to the top node. The incumbent will want to choose B. Thus, the unique SPNE for this game is

$$\{(N, E); [B, (H, H), (L, H)]\}$$

Remember that the SPNE refers to the set of strategies - not to the payoff to which it corresponds.

Problem 6

Consider the following game. (A, A) is the only pure strategy Nash Equilibrium (NE) of this game. Although (B, B) is a Pareto improvement compared with the NE, it is not sustainable. Consider instead the case where this game is repeated an infinite number of times, and players both have a time discount factor β. Also, consider the following "threat": "I will play B as long as you play B. Once you play A, I will play A ever after." Find the condition for β so that (B, B) can be sustained as a NE given the threat. (Hint: Consider a deviation from (B, B) in the first period.)

		Player 2: A	Player 2: B
Player 1	A	$(-2, -2)$	$(0, -8)$
	B	$(-8, 0)$	$(-1, -1)$

SOLUTION: For this deal to be an NE, it has to be the case that both players want to cooperate, given that the other player will respect the deal. Because the problem is exactly symmetric, both players will face the same decision between cooperation and betrayal, and we only have to consider that decision once to capture both players' perspectives. For either player, the payoff to cooperation will be to get -1 in every period:

$$\begin{aligned} u_i(cooperate) &= -1 + \beta * -1 + \beta^2 * -1 + ... \\ &= -1(1 + \beta + \beta^2 + ...) \\ &= \frac{-1}{1-\beta} \end{aligned}$$

where the last step employs the geometric series formula. If either player decides not to cooperate, in the period they make the betrayal, they would optimally want to play A and get a payoff of 0. In every subsequent period, the other player will play A, as per the deal, so the betrayer will want to also play A in every subsequent period, and get a payoff of -2. Thus, the utility of betrayal will be:

$$\begin{aligned} u_i(betray) &= 0 + \beta * -2 + \beta^2 * -2 + ... \\ &= -2\beta(1 + \beta + \beta^2 + ...) \\ &= \frac{-2\beta}{1-\beta} \end{aligned}$$

For cooperation to be an NE, we need that the payoff to cooperation is at least as good as the payoff to betrayal:

$$\begin{aligned} \frac{-1}{1-\beta} &\geq \frac{-2\beta}{1-\beta} \\ \Rightarrow \beta &\geq \frac{1}{2} \end{aligned}$$

Thus, the cooperative deal is an NE so long as the discount factor is at least one half.

Problem 7 - Bonus (5 pts on this PSet)

You and a friend have 1 brownie leftover after a party, and decide to split it. Specifically, you decide to split it via a sequential game - Player 1 cuts the brownie into two pieces and then Player 2 picks between the two pieces. Assume that both players' goal is to maximize the amount of brownie that they end up with (can define utility as the share of brownie received). Also assume that Player 1 is an extremely precise slicer - they can divide the brownie in exactly the manner they intend to.

(a) Describe the strategy sets of the two players.

SOLUTION: P1 divides the brownie and can do so in precisely the manner they intend. We can represent this as picking the proportions the brownie will be divided into. By definition, the "first" piece will constitute a proportion $p_1 \in [0, 1]$ and the "second" piece must then constitute the proportion $p_2 = (1 - p_1)$. Thus, we can represent P1's strategy set as selecting a pair of sizes for the brownie pieces:

$$S_1 = \{p_1 \in [0, 1], p_2 \in [0, 1]\} \; s.t. \; p_2 = 1 - p_1$$

P2 then chooses between the two pieces that P1 has divided, so P2's strategy set is simply:

$$S_2 = \{p_1, p_2\} = \{p_1, 1 - p_1\}$$

(b) Find the NE for this game.

To find an SPNE, we can use backwards induction. Start with P2. There are three cases:

- $p_1 > p_2 \Rightarrow$ P2 picks p_1
- $p_2 > p_1 \Rightarrow$ P2 picks p_2
- $p_2 = p_1 \Rightarrow$ P2 is indifferent and willing to play any pure or mixed strategy

Now we go back to P1. From the first case, we know that picking $p_1 > 1/2$ means that P2 will pick p_1, so P1 will end up with $p_2 = 1 - p_1 < 1/2$. From the second case, we know that picking $p_1 < 1/2$ means that P2 will pick p_2, so P1 will end up with $p_1 < 1/2$. Finally, from the third case, if P1 picks $p_1 = 1/2$, P2 is indifferent. But, for any possible strategy P2 might pick, P1 will be guaranteed to end up with a piece that has size 1/2, because both pieces are this size. Thus, P1 strictly prefers the third case to either of the first two cases. Thus, the set of SPNE occurs where P1 picks $p_1 = 1/2$ and P2 plays any strategy at all.

(c) Why would using this game be something that a Social Planner who values "fairness" like?

SOLUTION: From the SPNE described above, both players will ultimately end up with a piece of size 1/2. For a SP who likes "fairness" this sounds optimal. Specifically, the reason this outcome is achieved by this game is that the sequential nature of the game makes it such that P1, who does the actual cutting, is incentivized to cut into exact halves in order to maximize their own payout - cutting into anything other than exact halves will strictly reduce P1's payout.