Econ 201 Section 5 - Problem Set 5 Solutions

Problem 1

Suppose there are two firms that sell newspapers in the market: ATN and PGM. The aggregate demand for newspapers in the market is given by

$$p = 12 - y_A - y_P$$

where y_A refers to ATN quantity and y_P refers to PGM quantity. Further suppose that the total cost functions for the two firms are given by:

$$TC^A(y_A) = y_A^2$$
$$TC^P(y_P) = \frac{1}{2}y_P^2$$

(a) Suppose the firms engage in Cournot competition. Find the Nash Equilibrium quantities produced by each firm, as well as the aggregate quantity of newspapers and each firm's profit.

SOLUTION: To find the Cournot equilibrium, we want to find the best response functions for each firm and set them equal. Finding the best response functions is equivalent to solving each firm's problem, taking the quantity of the other firm as given:

$$\max_{y_A}(12 - y_A - y_P)y_A - y_A^2$$

$$[y_A]: \; 12 - y_A - y_P - y_A - 2y_A = 0$$
$$\Rightarrow y_A(y_P) = 3 - \frac{y_P}{4}$$

$$\max_{y_P}(12 - y_A - y_P)y_P - \frac{1}{2}y_P^2$$

$$[y_P]: \; 12 - y_A - y_P - y_P - y_P = 0$$
$$\Rightarrow y_P(y_A) = 4 - \frac{y_A}{3}$$

We then plug these into one another to find the collusive quantities:

$$y_P = 4 - \frac{3 - \frac{y_P}{4}}{3}$$
$$\Rightarrow y_p = \frac{36}{13}$$
$$\Rightarrow y_A = \frac{30}{13}$$

Aggregate quantity will simply be the sum of the quantities from each firm:

$$Y = \frac{66}{13}$$

And finally we can get profit by plugging the quantities into our profit functions:

$$\pi_A = (12 - y_A - y_P)y_A - y_A^2 = \frac{1800}{169}$$
$$\pi_P = (12 - y_A - y_P)y_P - \frac{1}{2}y_P^2 = \frac{2592}{169}$$

(b) Suppose the two firms now consider a deal. In the deal, the two firms will agree to each produce an identical quantity of newspapers. This quantity will be set so as to maximize industry profits (the objective of the maximization problem will be overall profit). What will this quantity be set to?

SOLUTION: Note that, unlike in class, this is slightly different from the monopoly problem, because we have two different cost functions, but are imposing that each firm produce the same amount. Thus, the maximization problem in question here will be:

$$\max_y (12 - 2y)2y - y^2 - \frac{1}{2}y^2$$

$$\max_y (12 - 2y)2y - \frac{3}{2}y^2$$

$$[y]: \; 24 - 8y - 3y = 0$$
$$\Rightarrow y = \frac{24}{11}$$

(c) Will both firms be willing to agree to the deal? Why or why not? Does this make intuitive sense? Why or why not?

SOLUTION: Keeping with what we did in class, we could plug the above quantity into each firm's best response function and see that both firms will want to deviate, and so neither firm would uphold the deal in a single (or finitely repeated) period game.

However, given the specific circumstances of this game, that we have two different cost functions, we can also show that in any kind of infinitely repeated game, PGM would not agree to this deal because it would reduce the amount of profit they get. We solved in part a) that PGM gets a profit of $\frac{2592}{169}$. Plugging the quantity from part b) into PGM's profit function tells us:

$$\begin{aligned}\pi_P &= (12-2y)y - \frac{1}{2}y^2 \\ &= (12 - 2\frac{24}{11})\frac{24}{11} - \frac{1}{2}(\frac{24}{11})^2 \\ \Rightarrow \pi_P &= \frac{1728}{121} < \frac{2592}{169}\end{aligned}$$

Unlike in class, PGM would not even prefer to collude without considering the possibility of betrayal at all. In class, we always looked at firms with identical technologies, so the incentive to collude always existed. Here, because PGM has "better" technology they actually do better just competing with ATN. If we repeated the process above with ATN, we'd see that ATN do have incentive to collude - it makes sense that the firm with "better" technology is the one that is happy to go it alone.

Problem 2

Consider the following Cournot model with three firms producing a homogenous good. Market demand is given by $Y = a - p$. Each firm faces a common constant marginal cost of c.

(a) What is the output and profit level of each firm?

SOLUTION: We can use the general Cournot solution for N firms that we got in class here with an $N = 3$ and a $b = 1$. Plugging into that, we get:

$$y_i = \frac{a-c}{(3+1)*1} = \frac{a-c}{4}$$

for all three firms. Then we can plug into any of the firms' profit function:

$$\begin{aligned}\pi_i &= (a - \frac{3(a-c)}{4})\frac{a-c}{4} - c\frac{a-c}{4} \\ &= (\frac{a-c}{4})^2\end{aligned}$$

(b) Suppose two of the firms merge. What is the output and profit level of each firm? Do the two firms have an incentive to merge?

SOLUTION: If, say, firms 1 and 2 merge, they will face the problem:

$$\max_{y_1,y_2}(a - y_1 - y_2 - y_3)(y_1 + y_2) - c(y_1 + y + 2)$$

$$[y_1] : a - y_1 - y_2 - y_3 - y_1 - y_2 - c = 0$$
$$[y_2] : a - y_1 - y_2 - y_3 - y_1 - y_2 - c = 0$$

Firm 3's problem is unchanged:

$$\max_{y_3}(a - y_1 - y_2 - y_3)y_3 - cy_3$$

$$[y_3] : a - y_1 - y_2 - y_3 - y_3 - c = 0$$

Solving the system of equations yields:

$$y_1 = y_2 = \frac{a-c}{6}$$
$$y_3 = \frac{a-c}{3}$$

We see that, perhaps unsurprisingly, this is the same as Cournot with two firms, with firms 1 and 2 collectively acting as one of the two firms, and firm three acting as the other one. Thus, collectively we know that firms 1 and 2 will earn a profit of $(\frac{a-c}{3})^2$. However from part a), we know that each firm makes over half this amount:

$$(\frac{a-c}{4})^2 = \frac{(a-c)^2}{16} > \frac{1}{2}(\frac{a-c}{3})^2 = \frac{(a-c)^2}{18}$$

Thus, the firms do not have incentive to merge, as they do better on their own.

Problem 3

Suppose there are two firms that sell houses in town - the Bluth Company and the Sitwell Company. The two firms sell differentiated houses - they are similar but not identical, and so act as substitutes in the market. Thus, the demand for either firm's houses will depend on the price set by the other firm. The demands and costs for the two products are:

$$y_b = 56 + 2p_s - 4p_b, \quad TC_b = 8y_b$$

$$y_s = 88 - 4p_s + 2p_b, \quad TC_s = 10y_s$$

(a) Suppose the two firms choose prices. What are their response functions?

SOLUTION: To get their response functions, we solve their profit maximization problems:

$$max_{p_b}(56 + 2p_s - 4p_b)p_b - 8(56 + 2p_s - 4p_b)$$

$$\begin{aligned} -8p_b + 88 + 2p_s &= 0 \\ \Rightarrow p_b &= 11 + \frac{1}{4}p_s \end{aligned}$$

$$max_{p_s}(88 - 4p_s + 2p_b)p_s - 10(88 - 4p_s + 2p_b)$$

$$\begin{aligned} -8p_s + 128 + 2p_b &= 0 \\ \Rightarrow p_s &= 16 + \frac{1}{4}p_b \end{aligned}$$

(b) Solve for the equilibrium prices, quantities and profits for both firms.

SOLUTION: In the equilibrium, it must be the case that neither firm has incentive to deviate from their current behavior, given the other firm's behavior. This is equivalent to the intersection of the best response functions. We can find this point by plugging either firm's best response into the other's. For instance, plugging the Sitwell best response into the Bluths' yields:

$$\begin{aligned} p_b &= 11 + \frac{1}{4}(16 + \frac{1}{4}p_b) \\ \Rightarrow p_b &= 16 \\ \Rightarrow p_s &= 20 \end{aligned}$$

From there, we plug back into our original demand functions to get quantities:

$$\begin{aligned} y_b &= 56 + 2(20) - 4(16) = 32 \\ y_s &= 88 - 4(20) + 2(16) = 40 \end{aligned}$$

And then plug prices and quantities into profit functions to finally get profits:

$$\begin{aligned} \pi_b &= (32)(16) - 8(32) = 256 \\ \pi_s &= (40)(20) - 10(40) = 400 \end{aligned}$$

Problem 4

In this question, we will formalize more the Bertrand model from class. We have two firms producing an identical good. Both firms have the same marginal cost $c > 0$ and they have no fixed costs. The firms compete by setting prices simultaneously (p_1 and p_2), and conditional on the chosen prices, quantity demanded from firm 1 is given by:

$$y_1(p_1, p_2) \begin{cases} 0 & \text{if } p_1 > p_2 \\ \frac{1}{2}\left(\frac{a-p_1}{b}\right) & \text{if } p_1 = p_2 \\ \frac{a-p_1}{b} & \text{if } p_1 < p_2 \end{cases}$$

Note that we would have a symmetric demand for firm 2 and when both firms charge the same price they equally split the market.

(a) Write down the set of strategies and payoffs for firm i.

SOLUTIONS: Either firm can set any non-negative price they would like:

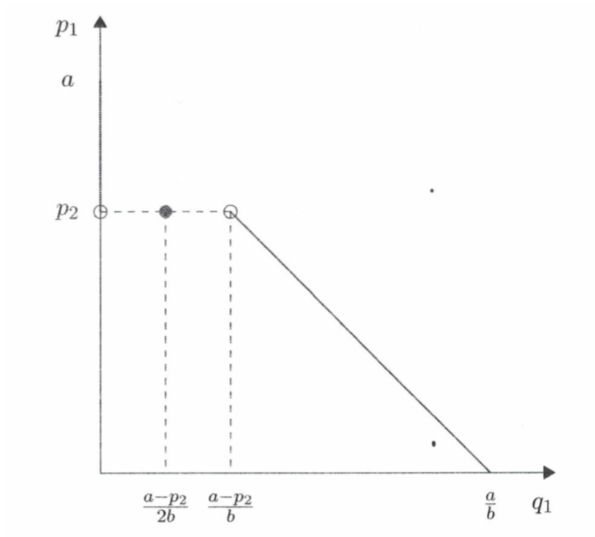
$$S_i = p_i \in [0, \infty) \ \forall \ i \in \{1, 2\}$$

From the given information on quantities sold, this will yield three "regions" of profit (payoffs) based on how the price relates to the price of the other firm. For any price above the other firm, there is 0 units sold and 0 profit. For a price equal to the other firm's, either firm gets profit for half of the market demand at that price. For a price less than the other firm's, the firm gets the profit for all of the market demand at that price:

$$\pi_i(p_i, p_{-i}) = \begin{cases} 0 \ if \ p_i > p_{-i} \\ \frac{1}{2}\left(\frac{a-p_i}{b}\right)(p_i - c) \ if \ p_i = p_{-i} \\ \left(\frac{a-p_i}{b}\right)(p_i - c) \ if \ p_i = p_{-i} \end{cases} \quad \forall \ i \in \{1, 2\}$$

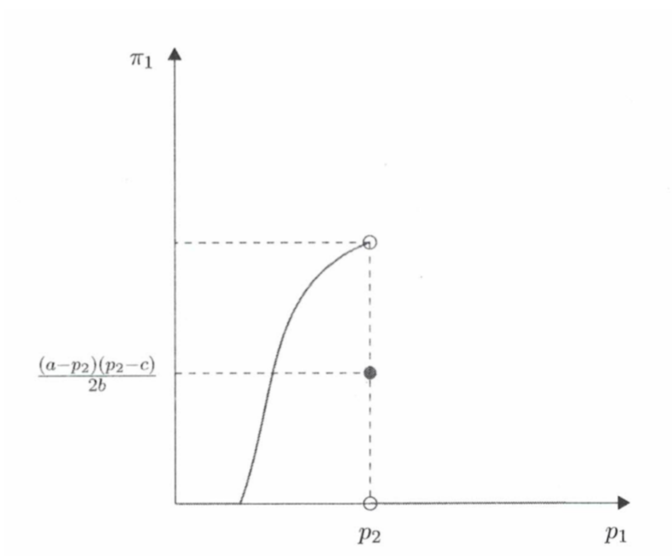
(b) For a given $p_2 < a$, draw the demand curve faced for firm 1.

SOLUTION: We can plot the demand that firm 1 will face from the given information in the problem:



(c) For a given $p_2 < a$, draw the payoff function for firm 1 (i.e., place π, payoff, on the y-axis and p on the x-axis).

SOLUTION: This time, we can plot this from the profit function that we developed in part a):



(d) Find the Nash equilibrium of this game by showing that the following cases are not a NE: (i) $p_1 > p_2 > c$, (ii) $p_1 > p_2, p_2 < c$, (iii) $p_1 = p_2 > c$, and (iv) $p_1 = p_2 < c$.

SOLUTION: Taking them in turn:

- (i): From our profit function, firm 1 is getting 0 profit and can strictly increase its profit by moving anywhere in the region $p_1 \in (c, p_2)$. Thus firm 1 has incentive to deviate and this is not an NE.
- (ii): If firm 2 is selling at a price below marginal cost, it is earning negative profit. Thus, firm 2 can do strictly better by selling no units and earning 0 profit. Thus, firm 2 has incentive to deviate and this is not an NE.
- (iii): Firm 1 can pick a price lower than p_2, $p_2 - \varepsilon$ and capture the whole market. So long as they pick a sufficiently small ε, this will increase their profit. Specifically, they would need to satisfy:

$$(\frac{a-(p_2-\varepsilon)}{b})(p_2-\varepsilon-c) > \frac{1}{2}(\frac{a-p_2}{b})(p_2-c)$$
$$\Rightarrow (\frac{a-p_2}{b})(p_2-c) > \varepsilon + \frac{\varepsilon^2}{b} - \frac{\varepsilon}{b}(p-c)$$

  and such an ε can always be found, so Firm 1 can increase their profit and has incentive to deviate. By symmetry, Firm 2 also has incentive to deviate, and so this is also not an NE.
- (iv): In this case, both firms are making negative profits. Thus, either firm has incentive to deviate to producing 0 units, making 0 profit. Thus, this is also not an NE.

None of these four cases are NEs, nor are the obvious analogues of $p_2 > p_1 > c$, $p_2 > p_1, p_1 < c$, $c > p_2 > p_1$, or $c > p_1 > p_2$. This covers every other possible case, and none of them are NEs, so $p_1 = p_2 = c$ is indeed the unique NE of this model.

Problem 5

We have two firms in the market. Firm 1 is the leader and chooses its quantity first (y_1). Firm 2 is the follower and chooses its quantity (y_2) after observing Firm 1's choice. The inverse demand function is: $p(Y) = a - bY$, where $Y = y_1 + y_2$. The firms have the following cost functions: $C_1(y_1) = c_1 y_1$ and $C_2(y_2) = c_2 y_2$.

(a) Using backward induction, solve for the equilibrium quantities of this game (i.e., y_1^{S*} and y_2^{S*}). You can assume that the SOCs are satisfied.

SOLUTION: Since we're using backwards induction, we solve the follower's problem first:

$$\max_{y_2} y_2(a - by_1 - by_2) - c_2 y_2$$

$$[y_2]: \; a - by_1 - by_2 - by_2 - c_2 = 0$$
$$\Rightarrow y_2 = \frac{a - by_1 - c_2}{2b}$$

Following the logic of backwards induction, the leader is able to anticipate what the follower will do. Thus, we solve the leader's problem next, plugging the follower's behavior directly into the maximization problem:

$$\max_{y_1} y_1(a - by_1 - b(\frac{a - by_1 - c_2}{2b})) - c_1 y_1$$

$$[y_1]: \; a - by_1 - \frac{a - by_1 - c_2}{2} - by_1 + \frac{b}{2}y_1 - c_1 = 0$$
$$\Rightarrow y_1^* = \frac{a - 2c_1 + c_2}{2b}$$

We can then plug this value of y_1 back into the follower's response function from earlier to finish with:

$$y_2^* = \frac{a + 2c_1 - 3c_2}{4b}$$

(b) Find Y^*, p^{S*}, π_1^{S*}, and π_2^{S*}.

SOLUTION: Adding the above quantities will give us the aggregate quantity:

$$Y^* = \frac{a - 2c_1 + c_2}{2b} + \frac{a + 2c_1 - 3c_2}{4b} = \frac{3a - 2c_1 - c_2}{4b}$$

Plug into demand to get the Stackleberg price:

$$p^* = a - b(\frac{3a - 2c_1 - c_2}{4b}) = a - \frac{3a - 2c_1 - c_2}{4}$$

And finally we can plug each firm's quantity along with the market price into each firm's profit function to get each firm's profit:

$$\begin{aligned}
\pi_1^* &= y_1^*(p^* - c_1) \\
&= \frac{a - 2c_1 + c_2}{2b}(a - \frac{3a - 6c_1 - c_2}{4}) \\
\pi_2^* &= y_2^*(p^* - c_2) \\
&= \frac{a + 2c_1 - 3c_2}{2b}(a - \frac{3a - 2c_1 - 5c_2}{4})
\end{aligned}$$

Problem 6

Suppose there are n firms, each with cost $c(y) = y$, playing the repeated Cournot quantity setting game. All firms discount future periods at rate $\beta \in (0, 1)$.

Assume the inverse market demand is $p(Y) = a - bY$. Use Y^m to denote the monopoly quantity. Consider the following profile of strategies: Each firm sets quantity $y_i = \frac{Y^m}{n}$ in the first period, and continue setting this quantity as long as nobody has deviated from it in the past. If any firm has deviated from this strategy, all firms play the Cournot Nash Equilibrium quantity in all future periods. We are interested in knowing when this profile of strategies constitute a Nash Equilibrium.

(a) What are the profits of each firm in the current period if all firms choose the Cournot Nash Equilibrium quantities?

SOLUTION: For any firm i, the problem will be:

$$\max_{y_i} \pi_i = (a - b\sum_{j=1}^{n} y_j)y_i - y_i$$

$$[y_i]: \; a - b\sum_{j=1}^{n} y_j - by_i - 1 = 0$$

$$\Rightarrow y_i = \frac{a - 1 - b\sum_{j\neq i} y_j}{2b}$$

Because all firms have identical problems, we can assume symmetry - in the equilibrium, all firms will produce the same quantity. Thus, we can say that $y_i^c = y_j^c \; \forall \; j \in \{1, ..., n\}$. Putting this into our above expression yields:

$$y_i = \frac{a - 1 - b(n-1)y_i}{2b}$$

$$\Rightarrow y_i^c = \frac{a-1}{(n+1)b}$$

We can plug this into our demand to find the equilibrium price:

$$p(Y^c) = a - bY^c$$

$$p(Y^c) = a - b\frac{n}{n+1}\frac{a-1}{b}$$

$$\Rightarrow p^c = \frac{a+n}{n+1}$$

And then we plug the price and quantity of any given firm into any of the firm's profit function to get each firm's profit:

$$\pi_i = y_i^c p^c - y_i^c$$

$$\pi_i = \frac{a-1}{(n+1)b}\frac{a+n}{n+1} - \frac{a-1}{(n+1)b}$$

$$\Rightarrow \pi_i^c = \frac{(a-1)^2}{(n+1)^2 b}$$

(b) If the firms collude to maximize collective profits in the current period, what are the profits of each firm?

SOLUTION: Colluding firms who are going to split profit equally will seek to set their collusive quantity so as to maximize the aggregate profit of the industry (thereby maximizing their share of total profit). Thus, we can solve this question by finding the monopolist profit for this industry and splitting that up amongst all n firms:

$$\max_{Y}(a - bY)Y - Y$$

$$[Y]: \; a - bY - bY - 1 = 0$$

$$\Rightarrow Y^M = \frac{a-1}{2b}$$

$$p^M = a - b\frac{a-1}{2b} = \frac{a+1}{2}$$

$$\pi^M = \frac{a+1}{2}\frac{a-1}{2b} - \frac{a-1}{2b} = \frac{(a-1)^2}{4b}$$

So, if the firms split this aggregate profit equally, they each end up with

$$\pi_i^{coll} = \frac{(a-1)^2}{4nb}$$

and we can notice that for any $n > 1$, this is greater than the Cournot profit from above, so we confirm that firms, in principal, have incentive to collude.

(c) If the other $n-1$ firms in the market were all choosing the collusive level of output, what quantity of output maximizes the profits of the nth firm in the current period? What are the profits this firm would receive in the current period?

SOLUTION: In part a), we found any given firm's best response to what the other firms were doing. Assuming the colluding firms split production evenly, we can plug $\frac{1}{n}$ of the monopoly quantity from part b) into this expression to find the optimal response to the other firms following the collusive level:

$$y_i^{BR} = \frac{a - 1 - b\sum_{j\neq i} y_j^{coll}}{2b}$$

$$= \frac{a - 1 - b(n-1)\frac{a-1}{2nb}}{2b}$$

$$\Rightarrow y_i^{BR} = \frac{na + a - n - 1}{4nb}$$

We can then plug both our deviating firm's quantity and every other firm's collusive quantity into the profit function to find the payoff of this situation:

$$\pi_i = (a - b(\frac{na+a-n-1}{4nb} + (n-1)\frac{a-1}{2nb}))\frac{na+a-n-1}{4nb} - \frac{na+a-n-1}{4nb}$$

$$\Rightarrow \pi_i^{betray} = \frac{3(n-1)(n+1)(a-1)^2}{16n^2b}$$

(d) Based on your answers to part (a)–(c), determine the level of patience (β) necessary to support the above strategies as a Nash Equilibrium.

SOLUTION: To show that the strategy is an NE, we need it to be the case that, if everyone else follows the deal, following the deal is weakly better than violating the deal. Thus, we need the expected utilities of cooperation and betrayal. When the firm cooperates, they simply get the collusive profit in every period:

$$\begin{aligned} u_i(coop) &= \pi_i^{coll} + \beta\pi_i^{coll} + \beta^2\pi_i^{coll}... \\ &= \frac{\pi_i^{coll}}{1-\beta} \end{aligned}$$

When we betray the deal, we'll optimally get the profit from part c), because that's the best response to everyone else following the deal. After that, we get Cournot every time:

$$\begin{aligned} u_i(betray) &= \pi_i^{betray} + \beta\pi_i^c + \beta^2\pi_i^c... \\ &= \pi_i^{betray} + \beta\pi_i^c(1+\beta+\beta^2+...) \\ &= \pi_i^{betray} + \frac{\beta\pi_i^c}{1-\beta} \end{aligned}$$

Finally, we just compare these two to see when it's better to cooperate, creating the circumstances for a collusive NE:

$$\begin{aligned} u_i(coop) &\geq u_i(betray) \\ \frac{\pi_i^{coll}}{1-\beta} &\geq \pi_i^{betray} + \frac{\beta\pi_i^c}{1-\beta} \\ \Rightarrow \beta &\geq \frac{\pi_i^{betray} - \pi_i^{coll}}{\pi_i^{betray} - \pi_i^c} \end{aligned}$$

Plugging in all of the profits we've calculated from previous sections will yield the expression:

$$\beta \geq \frac{\frac{3(n-1)(n+1)(a-1)^2}{16n^2b} - \frac{(a-1)^2}{4nb}}{\frac{3(n-1)(n+1)(a-1)^2}{16n^2b} - \frac{(a-1)^2}{(n+1)^2b}}$$

(e) How does your answer to part (d) vary with n? What does this tell you about how easy or difficult it is to support collusion when the number of firms increases?

SOLUTION: The RHS of our condition above is increasing in n. (This might be slightly non-obvious, but we can see that in both the numerator and the denominator we have the same first term having another term subtracted from it. The subtracted term is shrinking with n faster in the denominator. Thus, the numerator will rise relative to the denominator with n, which is equivalent to the whole expression increasing.) Thus, the level of patience needed is increasing in n - it is harder to support collusion when we have more firms.