Solution to Pset 6

1

a

The fruit farmer solves:

$$\max_{a,F} Fp_F - TC_F = F - 0.5F^2 - (a-2)^2$$

FOCs are:

$$[F]: \;\; 1 - F = 0 \Rightarrow F^* = 1$$

$$[a]: \quad -2(a-2) = 0 \Rightarrow a^* = 2$$

Profits are

$$(1)(1) - 0.5(1)^2 - (2-2)^2 = 0.5$$

b

The honey producer solves:

$$\max_H Hp_H - TC_H = H - 0.5H^2 - (2-a)H$$

The FOC is:

$$[H]: \quad 1 - H - (2-a) = 0 \Rightarrow H^* = 1$$

since the fruit farmer chose $a^* = 2$. Profits are also 0.5.

c

To find the jointly efficient solution, we solve:

$$\max_{a,F,H} \pi_F + \pi_H = F - 0.5F^2 - (a-2)^2 + H - 0.5H^2 - (2-a)H$$

FOCs are:

$$[a]: \quad -2(a-2) + H = 0 \Rightarrow H = 2(a-2)$$

$$[F]: \quad 1 - F = 0 \Rightarrow F^* = 1$$

$$[H]: \quad 1 - H - (2-a) = 0 \Rightarrow H = 1 + (a-2)$$

Setting the expressions in the first and third equations equal to one another, we get $a - 2 = 1 \Rightarrow a^* = 3$. Substituting this back in, we get $H^* = 2$. Joint profits are now

$$1 - 0.5 - 1^2 + 2 - 2 - (-1)2 = 1.5 > 1 = 0.5 + 0.5$$

d

While the fruit farm produces the same amount in either case, it now uses more fertilizer when owned by the social planner. This incentivizes the honey producer (also owned by the planner) to make more honey, and joint profits rise. The difference is because the fruit farmer acting alone does not consider the positive externality that the honey produce gets, but the social planner does.

Problem 2

We present here a negative externality problem that is framed a bit differently compared to the one we did in class, but we can still use the same analytical steps. A natural gas company is closely located near a lake where a boat rental store is also located. The gas company faces the following inverse demand curve: $p_g = 200 - g$. The plant's cost function is $C_g = 2g^2$. Similarly, the boat rental store faces the following inverse demand function: $p_B = 100 - B - 2g$. In words, as the gas company produces more g, the demand for boat rentals decreases (because pollution makes a boat ride less pleasant). The store's cost function is given by: $C_B = 0.5B^2$. Throughout this question, you can assume that SOCs are satisfied.

(a) Solve the private profit maximization problems for the gas company and the boat rental. For both businesses, calculate the optimal quantities, prices and profit levels.

SOLUTION: For the gas company, the problem is:

$$\max_g (200 - g)g - 2g^2$$

$$[g] : 200 - g - g - 4g = 0$$

$$\Rightarrow g^* = \frac{500}{3}$$

The boat rental's problem is:

$$\max_B (100 - B - 2g)B - 0.5B^2$$

$$[b] : 100 - B - 2g - B - B = 0$$

$$\Rightarrow B^* = \frac{100 - 2g}{3}$$

We can plug into demand functions to get the prices:

$$p_g = 200 - \frac{100}{3}$$

$$\Rightarrow p_g^* = \frac{500}{3}$$

$$p_B = 100 - \frac{100 - 2\frac{100}{3}}{3} - 2\frac{100}{3}$$

$$\Rightarrow p_B = \frac{200}{9}$$

And finally, we can plug into profit functions to calculate equilibrium profits:

$$\pi_g = \frac{500}{3}\frac{100}{3} - 2(\frac{100}{3})^2$$
$$\pi_g^* = \frac{10,000}{3}$$
$$\pi_B = \frac{200}{9}\frac{100 - 2\frac{100}{3}}{3} - 0.5(\frac{100 - 2\frac{100}{3}}{3})^2$$
$$\pi_B^* = \frac{15,000}{81}$$

(b) Solve for the socially efficient solution. In other words, assume that a social planner owns both businesses. Similar to part a), find quantities, prices, and the joint profit level. Specify how much each business contributes towards joint profits.

SOLUTION: For an SP who own's both, the problem is:

$$\max_{g,B}(200 - g)g - 2g^2 + (100 - B - 2g)B - 0.5B^2$$

$$[g] : \ 200 - 2g - 4g - 2B = 0$$
$$[B] \ 100 - 2B - 2S - B = 0$$

Solving the system of equations yields:

$$g^{sp} = \frac{200}{7}$$
$$B^{sp} = \frac{100}{7}$$

And then plugging into demands and profits, as above, will give us:

$$p_g^{sp} = \frac{1200}{7}$$
$$p_B^{sp} = \frac{200}{7}$$
$$pi_g^{sp} = \frac{160,000}{49}$$
$$pi_B^{sp} = \frac{15,000}{81}$$

(c) Compare the firm/joint profit levels between parts a) and b).

SOLUTION: Comparing results from parts a) and b), we see that the gas company earns less profit and the boat rental earns more. Comparing the aggregate, we see:

$$\frac{160,000}{49} + \frac{15,000}{49} > \frac{270,000}{81} + \frac{15,000}{81}$$

so, although the gas producer earns less, the gain for the boat rental more than makes up for this. This is to be expected: the SP was trying to maximize aggregate profit, so we should have higher aggregate profit under the SP's solution than the market solution.

(d) You work for the environmental agency regulating pollution in the lake. If you want to implement a quota, on which business would you impose the quota, and at what level would you set it? In terms of quantities, prices and profit levels, for both firms, what are the new equilibrium levels? (Hint: you need no further calculations)

SOLUTION: The problem in the market was that the gas producer was producing too much, given that they were not taking the negative externality into account. Given that the gas producer wants to produce more than the optimal amount, if we create a maximum on the gas company's production at, $\frac{200}{7}$, the socially efficient amount, the gas company will produce that much, and we'll get the socially optimal solution.

(e) Instead, consider now imposing a Pigouvian tax on the steel plant. Solve the new profit maximization problem for the gas company, and find the equilibrium tax rate T^* that achieves the socially efficient outcome. (Hint: different to what we did in class, the tax rate T is imposed on units of g)

SOLUTION: With a per-unit tax, the gas company's new problem is:

$$\max_g (200 - g)g - 2g^2 - Tg$$

$$[g]: \ 200 - g - g - 4g - T = 0$$
$$T = 200 - 6g$$

If we want the socially efficient outcome, we can set g to the socially efficient amount to derive the optimal per-unit tax:

$$T = 200 - 6\frac{200}{7}$$
$$\Rightarrow T^P = \frac{200}{7}$$

(f) Finally, let's consider implementing property rights and trade. As the government agency you decide that the boat rental store has the right for a clean lake. Furthermore, you create a 'market' where the gas company can buy the 'right' to pollute from the boat rental store. Given that pollution is an unavoidable byproduct of producing g, essentially the gas company will be paying p_{gT} for the right to produce one unit of g. Without using any shortcuts, and by solving the profit maximization problems for both businesses, find p_{gT}^* and confirm that the equilibrium quantities and prices for both businesses are identical to those from the socially efficient solution.

SOLUTION: The new gas company problem is:

$$\max_g (200 - g)g - 2g^2 - p_{gT}g$$

$$\begin{aligned} [g]: \ 200 - 6g - p_{gT} &= 0 \\ \Rightarrow p_{gT} &= 200 - 6g \end{aligned}$$

The new boat rental problem is:

$$\max_g (100 - B - 2g)B - 0.5B^2 + p_{gT}g$$

$$\begin{aligned} [B]: \ 100 - 2B - B - 2g &= 0 \\ \Rightarrow B &= \frac{100 - 2g}{3} \\ [g]: \ -2B + p_{gT} &= 0 \\ \Rightarrow p_{gT} &= 2B \end{aligned}$$

Subbing back into the gas company's FOC yields:

$$2B = p_{gT} = 200 - 6g$$

We can notice that the above condition, along with the boat rental's first FOC, is identical to the system of equations, so the solution will occur at the same place - the socially efficient solution.

Question 3

(a) The profit maximization problem for H_1 is

$$\max_{H_1,F_1} \{4\ln(H_1) - H_1 + 2\ln(F) - F_1\}$$
$$\text{s.t.} \quad F = F_1 + F_2.$$

or

$$\max_{H_1,F_1} \{4\ln(H_1) - H_1 + 2\ln(F_1 + 0.5) - F_1\}$$

FOC's:

$$[H_1]: \quad \frac{4}{H_1} - 1 = 0 \Longrightarrow H_1^* = 4.$$
$$[F_1]: \quad \frac{2}{F_1 + 0.5} - 1 = 0 \Longrightarrow F_1^* = 1.5.$$

(b) From (a), we know that

$$\max_{H_1,F_1} \{4\ln(H_1) - H_1 + 2\ln(F_1 + F_2) - F_1\}$$

so the FOC's are

$$[H_1]: \quad \frac{4}{H_1} - 1 = 0 \Longrightarrow H_1^* = 4.$$
$$[F_1]: \quad \frac{2}{F_1 + F_2} - 1 = 0 \Longrightarrow F_1^*(F_2) = 2 - F_2.$$

For H_2,

$$\max_{H_2,F_2} \{4\ln(H_2) - H_2 + \ln(F_1 + F_2) - F_2\}$$

so the FOC's are

$$[H_2]: \quad \frac{4}{H_2} - 1 = 0 \Longrightarrow H_2^* = 4.$$
$$[F_2]: \quad \frac{1}{F_1 + F_2} - 1 = 0 \Longrightarrow F_2^*(F_1) = 1 - F_1.$$

(c) Usually, by solving simultaneously the best-response functions, we can find an equilibrium. Let's see what happens if we do this here.

$$F_1 = 2 - (1 - F_1)$$
$$0 = 1$$

which is not true. Mathematically, this does not work because we have a corner solutions, and thus the FOC does not hold with equality.

(d) See figure 1 below. The best-response function for H_1 is in red. For $F_2 \geq 2$, $F_1^* = 0$; thus, H_1 should not invest in flowers when $F_2 \geq 2$. The best-response function for H_2 is in blue. For $F_1 \geq 1$, $F_2^* = 0$; thus H_2 should not invest in flowers when $F_1 \geq 1$.

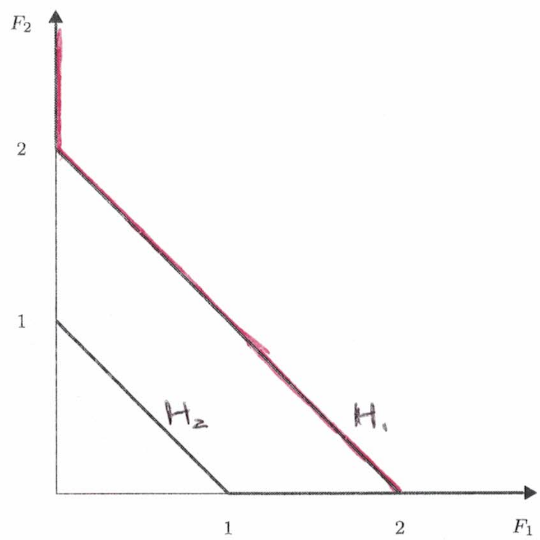

Figure 1: Plot of the best response functions.

The equilibrium is where the two functions cross ($F_1^* = 2, F_2^* = 0$). If you look at the profit functions carefully, you'll see that H_1 benefits the most (puts higher value) from the flowers. Thus, in equilibrium, only H_1 invests and H_2 will simply free-ride, still benefiting from the flowers.

(e)

$$\max_{H_1,F_1,H_2,F_2} \{4\ln(H_1) - H_1 + 2\ln(F_1 + F_2) - F_1 + 4\ln(H_2) - H_2 + \ln(F_1 + F_2) - F_2\}$$

or

$$\max_{H_1,F_1,H_2,F_2} \{4[\ln(H_1) + \ln(H_2)] - H_1 - H_2 + 3\ln(F_1 + F_2) - F_1 - F_2\}$$

$$\max_{H_1,H_2,F} \{4[\ln(H_1) + \ln(H_2)] - H_1 - H_2 + 3\ln(F) - F\}$$

We will just look at the FOC for F:

$$[F]: \quad \frac{3}{F} - 1 = 0 \Longrightarrow F^{E^*} = 3.$$

Hence, the private solution is smaller:

$$F_1^* + F_2^* = 2 < 3 = F^{E^*}$$

Question 4

(a) The utility maximization problem for individual 1 is given by

$$\max_{C_1, E_1} \quad 5\ln(C_1) + 2\ln(E)$$

$$\text{s.t.} \quad p_C C_1 + p_E E_1 = m_1$$

$$E_1 + E_2 = E$$

or

$$\max_{C_1, E_1} \quad 5\ln(C_1) + 2\ln(E_1 + E_2)$$

$$\text{s.t.} \quad 2C_1 + 3E_1 = 100$$

The Lagrangian is

$$\mathcal{L} = 5\ln(C_1) + 2\ln(E_1 + E_2) + \lambda[100 - 2C_1 - 3E_1]$$

FOC's:

$$[C_1]: \quad \frac{5}{C_1} - 2\lambda = 0 \Longrightarrow \frac{5}{C_1} = 2\lambda$$

$$[E_1]: \quad \frac{2}{E_1 + E_2} - 3\lambda = 0 \Longrightarrow \frac{2}{E_1 + E_2} = 3\lambda$$

$$[\lambda]: \quad 100 - 2C_1 - 3E_1 = 0.$$

Then $[E_1]/[C_1]$ implies

$$\frac{2}{5}\frac{C_1}{E_1 + E_2} = \frac{3}{2} \Longrightarrow C_1 = \frac{15}{4}(E_1 + E_2).$$

Plug into $[\lambda]$ to get

$$100 - 2\left[\frac{15}{4}(E_1 + E_2)\right] - 3E_1 = 0.$$

Solving this equation, we get

$$E_1 = \frac{200}{21} - \frac{5}{7}E_2.$$

Since individuals have symmetric utilities,

$$C_2 = \frac{15}{4}(E_1 + E_2)$$

and

$$E_2 = \frac{200}{21} - \frac{5}{7}E_1,$$

which is equivalent to saying $E_1 = E_2 = \bar{E}$ and $C_1 = C_2 = \bar{C}$. Then

$$\bar{E} = \frac{200}{21} - \frac{5}{7}\bar{E}$$
$$\bar{E} = \frac{50}{9} = E_1^* = E_2^*$$

and

$$E^* = E_1^* + E_2^* = \frac{100}{9}.$$

Plugging into the equations for C,

$$\bar{C} = \frac{15}{4}\frac{100}{9} = \frac{125}{3}$$

so that

$$C_1^* = C_2* = \frac{125}{3}.$$

(b) To find the socially efficient solution, we solve

$$\max_{C_1,E_1,C_2,E_2} \quad 5\ln(C_1) + 5\ln(C_2) + 4\ln(E)$$

$$\text{s.t.} \quad p_C C_1 + p_E E_1 + p_C C_2 + p_E E_2 = m_1 + m_2$$

which just amounts to solving the sum of utilities subject to an "economy wide" resource constraint. Note that this is equivalent to solving

$$\max_{C_1,C_2,E} \quad 5\ln(C_1) + 5\ln(C_2) + 4\ln(E)$$

$$\text{s.t.} \quad 2(C_1 + C_2) + 3E = 200$$

The Lagrangian is

$$\mathcal{L} = 5\ln(C_1) + 5\ln(C_2) + 4\ln(E) + \lambda[200 - 2(C_1 + C_2) - 3E]$$

FOC's:

$$[C_1]: \quad \frac{5}{C_1} - 2\lambda = 0 \Longrightarrow C_1 = \frac{5}{2\lambda}$$
$$[C_2]: \quad \frac{5}{C_2} - 2\lambda = 0 \Longrightarrow C_2 = \frac{5}{2\lambda}$$
$$[E]: \quad \frac{4}{E} - 3\lambda = 0 \Longrightarrow E = \frac{4}{3\lambda}$$
$$[\lambda]: \quad 200 - 2(C_1 + C_2) - 3E = 0$$

Plug $[C_1]$, $[C_2]$ and $[E]$ into $[\lambda]$ to get

$$2\left(\frac{5}{2\lambda} + \frac{5}{2\lambda}\right) + 3\frac{4}{3\lambda} = 200$$

$$\frac{10}{\lambda} + \frac{4}{\lambda} = 200 \Longrightarrow \lambda = \frac{7}{100}.$$

Now we just plug this back into the FOCs to get:

$$C_1^{E*} = C_2^{E*} = \frac{250}{7}$$

$$E^{E*} = \frac{400}{21}$$

(c) As expected, the previous results show that total investment in education is higher under the socially. Efficient solution. Similarly, individuals are better off under the socially efficient solution:

$$U_1(C_1^*, E^*) = U_2(C_2^*, E^*) = 5ln(\frac{125}{3}) + 2ln(\frac{100}{9}) \approx 23.5$$

$$U_1(C_1^{E*}, E^{E*}) = U_2(C_2^{E*}, E^{E*}) = 5ln(\frac{250}{7}) + 2ln(\frac{400}{21}) \approx 23.8$$